

Continuous Time and Consistent Histories

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We discuss the use of histories labelled by a continuous time in the approach to consistent-histories quantum theory in which propositions about the history of the system are represented by projection operators on a Hilbert space. This extends earlier work by two of us [1] where we showed how a continuous time parameter leads to a history algebra that is isomorphic to the canonical algebra of a quantum *field* theory. We describe how the appropriate representation of the history algebra may be chosen by requiring the existence of projection operators that represent propositions about time average of the energy. We also show that the history description of quantum mechanics contains an operator corresponding to velocity that is quite distinct from the momentum operator. Finally, the discussion is extended to give a preliminary account of quantum field theory in this approach to the consistent histories formalism.

I. INTRODUCTION

The consistent-histories approach to quantum theory can be formulated in several different ways. In the original scheme [2-4], the crucial object is the decoherence function written as

$$d(\alpha, \beta) = \text{tr}(\tilde{C}_\alpha^\dagger \rho \tilde{C}_\beta) \quad (1)$$

where ρ is the initial density-matrix, and where the *class operator* \tilde{C}_α is defined in terms of the standard Schrödinger-picture projection operators α_{t_i} as

$$\tilde{C}_\alpha := U(t_0, t_1)\alpha_{t_1}U(t_1, t_2)\alpha_{t_2}\dots U(t_{n-1}, t_n)\alpha_{t_n}U(t_n, t_0), \quad (2)$$

where $U(t, t') = e^{-i(t-t')H/\hbar}$ is the unitary time-evolution operator from time t to t' . Each projection operator α_{t_i} represents a proposition about the system at time t_i , and the class operator \tilde{C}_α represents the composite history proposition “ α_{t_1} is true at time t_1 , and then α_{t_2} is true at time t_2 , and then \dots , and then α_{t_n} is true at time t_n ”.

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As a product of (generically, non-commuting) projection operators, the class-operator \tilde{C}_α is not itself a projector. This difference between the representation of propositions in standard quantum mechanics and in the history theory is avoided in the alternative approach [5,6] to the latter in which the history proposition “ α_{t_1} is true at time t_1 , and then α_{t_2} is true at time t_2 , and then \dots , and then α_{t_n} is true at time t_n ” is represented by the *tensor product* $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \dots \otimes \alpha_{t_n}$ which, unlike \tilde{C}_α , is a genuine projection operator. In this ‘history projection operator’ (HPO) scheme, the decoherence function can be written as

$$d(\alpha, \beta) = \text{tr}_{\mathcal{V} \otimes \mathcal{V}}(\alpha \otimes \beta X) \quad (3)$$

for a suitable operator X (independent of α and β) defined on $\mathcal{V} \otimes \mathcal{V}$ [7]. Here, \mathcal{V} denotes the ‘history space’ $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \dots \otimes \mathcal{H}_{t_n}$ on which the history-proposition projectors α and β are defined, where, for each t_i , \mathcal{H}_{t_i} is a copy of the Hilbert space \mathcal{H} of the standard quantum theory. The representation of history propositions with projection operators has clear advantages when discussing logical operations; in our case, quantum temporal logic. It could also be applicable in quantum gravity situations where there is no background concept of time and where, therefore, the construction of a class operator is particularly problematic: in this case, history propositions could still be represented by projection operators, but on a Hilbert space that does not arise as a temporal tensor product.

The introduction of a *continuous* time clearly poses difficulties for both approaches to the consistent history theory: in the class-operator scheme one has to define continuous products of projection operators; in the HPO approach, the problem is to define a continuous *tensor* product of projection operators.

In an earlier paper by two of us [1], the latter problem was tackled by exploiting the well-known existence of continuous tensor products of coherent states. However, several interesting issues were sidestepped in the process. For example, the natural history propositions in this scheme are represented by continuous tensor products of projectors onto coherent states, and these do not have a transparent physical interpretation. On the other hand, the formalism as given was not well-equipped to handle the more physically-motivated history propositions about continuous time averages.

In the present paper we take a fresh look at the question of continuous time in the HPO formalism. As in our earlier work, the starting point is the *history group*: a history-analogue of the canonical group used in standard quantum mechanics. The key idea is that a unitary representation of the history group leads to a self-adjoint representation of its Lie algebra, the spectral projectors of which are to be interpreted as propositions about the histories of the system. Thus we employ a history group whose associated projection operators represent propositions about continuous-time histories. It transpires that the history algebra for one-dimensional quantum mechanics is *infinite* dimensional—in fact, it is isomorphic to the canonical commutation algebra of a standard quantum *field* theory in one spatial dimension. This suggests that it might be profitable to study the history theory using tools that are normally employed in quantum field theory. This we shall do; in particular, we show that the physically appropriate representation of the history algebra can be selected by requiring the existence of operators that represent propositions about the time-averaged values of the energy. The Fock space thus constructed is related to the notion of a continuous tensor product as used in our earlier paper, thus establishing the link with the idea of continuous temporal logic. We also introduce the continuous-time history analogue of the Heisenberg picture, and we discuss the role of velocity in the history theory. Finally, the discussion is extended to include the case of a free relativistic quantum field.

In what follows we have deliberately adopted a ‘physicist’s approach’ to the analytical problems that arise in the theory; for example, we frequently use unsmeared commutation relations, and domains of operators are not discussed. This enables us to present the essential physical ideas without getting lost in mathematical detail. However, nothing of real importance is hidden thereby since only Fock space representations of the history algebra are used, and the full mathematical theory of these is well-known from normal quantum field theory and poses no major problems.

II. THE HISTORY SPACE

A. The History Group

We start by considering the HPO version of the quantum theory of a particle moving on the real line \mathbb{R} . As explained above, the history proposition “ α_{t_1} is true at time t_1 , and then α_{t_2} is true at time t_2 , and then \dots , and then α_{t_n} is true at time t_n ” is represented by the projection operator $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \dots \otimes \alpha_{t_n}$ on the n -fold tensor product $\mathcal{V}_n = \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \dots \otimes \mathcal{H}_{t_n}$ of n -copies of the Hilbert-space \mathcal{H} of the canonical theory. Since \mathcal{H} carries a representation of the Heisenberg-Weyl group with Lie algebra

$$[x, p] = i\hbar, \quad (4)$$

the Hilbert space \mathcal{V}_n carries a unitary representation of the n -fold product group whose generators satisfy

$$[x_k, x_m] = 0 \quad (5)$$

$$[p_k, p_m] = 0 \quad (6)$$

$$[x_k, p_m] = i\hbar\delta_{km} \quad (7)$$

with $k, m = 1, 2, \dots, n$. Thus the Hilbert space \mathcal{V}_n carries a representation of the ‘history group’ whose Lie algebra is defined to be that of Eqs. (5)–(7). However, we can also turn the argument around and *define* the history version of n -time quantum mechanics by starting with Eqs. (5)–(7). In this approach, \mathcal{V}_n arises as a representation space for Eqs. (5)–(7), and tensor products $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \dots \otimes \alpha_{t_n}$ that correspond to sequential histories about the values of position or momentum (or linear combinations of them) are then elements of the spectral representations of this Lie algebra.

We shall employ this approach to discuss continuous-time histories. Thus, motivated by Eqs. (5)–(7), we start with the history-group whose Lie algebra (referred to in what follows as the ‘canonical history algebra’, or CHA for short) is [1]

$$[x_{t_1}, x_{t_2}] = 0 \quad (8)$$

$$[p_{t_1}, p_{t_2}] = 0 \quad (9)$$

$$[x_{t_1}, p_{t_2}] = i\hbar\delta(t_1 - t_2) \quad (10)$$

where $-\infty \leq t_1, t_2 \leq \infty$. Note that these operators are in the *Schrödinger* picture: they must not be confused with the Heisenberg-picture operators $x(t), p(t)$ of normal quantum theory.

An important observation is that Eqs. (8)–(10) are mathematically the same as the canonical commutation relations of a quantum *field* theory in one space dimension:

$$[\phi(x_1), \phi(x_2)] = 0 \quad (11)$$

$$[\pi(x_1), \pi(x_2)] = 0 \quad (12)$$

$$[\phi(x_1), \pi(x_2)] = i\hbar\delta(x_1 - x_2). \quad (13)$$

This analogy will be exploited fully in the present paper. For example, the following two issues arise immediately. Firstly—to be mathematically well-defined—equations of the type Eqs. (8)–(10) must be smeared with test functions to give

$$[x_f, x_g] = 0 \quad (14)$$

$$[p_f, p_g] = 0 \quad (15)$$

$$[x_f, p_g] = i\hbar \int_{-\infty}^{\infty} f(t)g(t) dt, \quad (16)$$

which leads at once to the question of which class τ of test functions to use. The minimal requirement for the right hand side of Eq. (16) to make sense is that τ must be a linear subspace of the space $L^2(\mathbb{R}, dt)$ of square integrable functions on \mathbb{R} . For the moment we shall leave τ unspecified beyond this.

The second issue is concerned with finding the physically appropriate representation of the CHA Eqs. (14)–(16), bearing in mind that infinitely many unitarily inequivalent representations are known to exist in the analogous case of Eqs. (11)–(13). Note that this problem does not arise in standard quantum mechanics, or in the history version of quantum mechanics with propositions defined at a finite number of times, since—by the Stone-von Neumann theorem—there is a unique representation of the corresponding algebra up to unitarily equivalence.

Of course, from the perspective of the history theory the physically appropriate representation is expected to involve some type of continuous tensor product; this was the path followed in our earlier work [1]. On the other hand, in standard quantum field theory there is a folk lore, going back at least to a famous paper by Araki [8], to the effect that requiring the Hamiltonian to exist as a proper self-adjoint operator is sufficient to select a unique representation; for example, the representations appropriate for a free boson field with different masses are unitarily inequivalent. In our case, this suggests that the appropriate representation of the algebra Eqs. (14)–(16) should be chosen by requiring the existence of operators that represent history propositions about (time-averaged) values of the energy. As we shall see, this is indeed the case.

B. The Hamiltonian Algebra

We start with the ubiquitous example of the one-dimensional, simple harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2. \quad (17)$$

The naïve idea behind the HPO theory is that to each time t there is associated a Hilbert space \mathcal{H}_t that carries propositions appropriate to that time (the ‘naïvety’ refers to the fact that, in a *continuous* tensor product $\otimes_{t \in \mathbb{R}} \mathcal{H}_t$, the individual Hilbert spaces \mathcal{H}_t do not strictly exist as subspaces; this is related to the need to smear operators). Thus we expect to have a one-parameter family of operators

$$H_t := \frac{p_t^2}{2m} + \frac{m\omega^2}{2}x_t^2 \quad (18)$$

that represent the energy at time t .

As it stands, the right hand side of Eq. (18) is not well-defined, just as in normal canonical quantum field theory it is not possible to define products of field operators at the same spatial point. However, the commutators of H_t with the generators of the CHA can be computed formally as

$$[H_t, x_s] = -\frac{i\hbar}{m}\delta(t-s)p_s \quad (19)$$

$$[H_t, p_s] = i\hbar m\omega^2\delta(t-s)x_s \quad (20)$$

$$[H_t, H_s] = 0 \quad (21)$$

and are the continuous-time, history analogues of the familiar result in standard quantum theory:

$$[H, x] = -\frac{i\hbar}{m}p \quad (22)$$

$$[H, p] = i\hbar m\omega^2x. \quad (23)$$

In standard quantum theory, the spectrum of the Hamiltonian operator can be computed directly from the algebra of Eqs. (22)–(23) augmented with the requirement that the underlying representation of the canonical commutation relations Eq. (4) is irreducible. This suggests that we try to *define* the history theory by requiring the existence of a family of operators H_t that satisfy the relations Eqs. (19)–(21) and where the representation of the canonical history algebra Eqs. (8)–(10) is irreducible. More precisely, we augment the CHA with the algebra (in semi-smearred form)

$$[H(\chi), x_t] = -\frac{i\hbar}{m}\chi(t)p_t \quad (24)$$

$$[H(\chi), p_t] = i\hbar m\omega^2\chi(t)x_t \quad (25)$$

$$[H(\chi_1), H(\chi_2)] = 0 \quad (26)$$

where $H(\chi)$ is the history energy-operator, time averaged with the function χ ; heuristically, $H(\chi) = \int_{-\infty}^{\infty} dt \chi(t)H_t$.

It is useful to integrate these equations in the following sense. If self-adjoint operators $H(\chi)$ exist satisfying Eqs. (24)–(26), we can form the unitary operators $e^{iH(\chi)/\hbar}$, and these satisfy

$$e^{iH(\chi)/\hbar} x_t e^{-iH(\chi)/\hbar} = \cos[\omega\chi(t)]x_t + \frac{1}{m\omega} \sin[\omega\chi(t)]p_t \quad (27)$$

$$e^{iH(\chi)/\hbar} p_t e^{-iH(\chi)/\hbar} = -m\omega \sin[\omega\chi(t)]x_t + \cos[\omega\chi(t)]p_t. \quad (28)$$

However, it is clear that the right hand side of Eqs. (27)–(28) defines an *automorphism* of the canonical history algebra Eqs. (8)–(10). Thus the task in hand can be rephrased as that of finding an irreducible representation of the CHA in which these automorphisms are unitarily implementable: the self-adjoint generators of the corresponding unitary operators will then be the desired time-averaged energy operators $H(\chi)$ [strictly speaking, weak continuity is also necessary but this poses no additional problems in the cases of interest here].

C. The Fock Representation

It is natural to contemplate the use of a Fock representation of the CHA since this plays such a central role in the analogue of a free quantum field in one spatial dimension. To this end, we start by defining the ‘annihilation operator’

$$b_t := \sqrt{\frac{m\omega}{2\hbar}} x_t + i\sqrt{\frac{1}{2m\omega\hbar}} p_t \quad (29)$$

in terms of which the CHA (8)–(10) becomes

$$[b_t, b_s] = 0 \quad (30)$$

$$[b_t, b_s^\dagger] = \delta(t - s). \quad (31)$$

Note that

$$\hbar\omega b_t^\dagger b_s = \frac{1}{2m} p_t p_s + \frac{m\omega^2}{2} x_t x_s - \frac{\hbar\omega}{2} \delta(t - s) \quad (32)$$

which suggests that there exists an additively renormalised version of the operator H_t in Eq. (18) of the form $\hbar\omega b_t^\dagger b_t$. In turn, this suggests strongly that a Fock space based on Eq. (29) should provide the operators we seek.

To make this explicit we recall that the bosonic Fock space $\mathcal{F}[\mathcal{H}]$ associated with a Hilbert space \mathcal{H} is defined as

$$\mathcal{F}[\mathcal{H}] := \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes_S \mathcal{H}) \oplus \cdots \quad (33)$$

where $\mathcal{H} \otimes_S \mathcal{H}$ denotes the symmetrised tensor product of \mathcal{H} with itself. Any unitary operator U on the ‘one-particle’ space \mathcal{H} gives a unitary operator $\Gamma(U)$ on $\mathcal{F}[\mathcal{H}]$ defined by

$$\Gamma(U) := 1 \oplus U \oplus (U \otimes U) \oplus \cdots \quad (34)$$

Furthermore, if $U = e^{iA}$ for some self-adjoint operator A on \mathcal{H} , then $\Gamma(U) = e^{i d\Gamma(A)}$ where

$$d\Gamma(A) := 0 \oplus A \oplus (A \otimes 1 + 1 \otimes A) \oplus \cdots \quad (35)$$

The implications for us of these well-known constructions are as follows. Consider the Fock space $\mathcal{F}[L^2(\mathbb{R}, dt)]$ that is associated with the Hilbert space $L^2(\mathbb{R}, dt)$ via the annihilation operator b_t defined in Eq. (29); *i.e.*, the space built by acting with (suitably smeared) operators b_t^\dagger on the ‘vacuum state’ $|0\rangle$ that satisfies $b_t|0\rangle = 0$ for all $t \in \mathbb{R}$. The equations Eq. (27)–(28) show that, *if* it exists, the operator $e^{iH(\chi)/\hbar}$ acts on the putative annihilation operator b_t as

$$e^{iH(\chi)/\hbar} b_t e^{-iH(\chi)/\hbar} = e^{-i\omega\chi(t)} b_t. \quad (36)$$

However, thought of as an action on $L^2(\mathbb{R}, dt)$, the operator $U(\chi)$ defined by

$$(U(\chi)\psi)(t) := e^{-i\omega\chi(t)} \psi(t) \quad (37)$$

is unitary for any measurable function χ . Hence, using the result mentioned above, it follows that in this particular Fock representation of the CHA the automorphism on the right hand side of Eq. (36) *is* unitarily implementable, and hence the desired self-adjoint operators exist. Note that $H(\chi) = \hbar\omega d\Gamma(\hat{\chi})$, where the self-adjoint operator $\hat{\chi}$ is defined on $L^2(\mathbb{R}, dt)$ as

$$(\hat{\chi}\psi)(t) := \chi(t)\psi(t). \quad (38)$$

In summary, we have shown that the Fock representation of the CHA Eqs. (8)–(10) associated with the annihilation operator b_t of Eq. (29) is such that there exists a family of self-adjoint operators $H(\chi)$ for which the algebra Eqs. (24)–(26) is satisfied. This Fock space is the desired carrier of the history propositions in our theory. Note that, in this case, the natural choice for the test function space $\tau \subseteq L^2(\mathbb{R}, dt)$ used in Eqs. (14)–(16) is simply $L^2(\mathbb{R}, dt)$ itself.

The position history-variable x_t can be written in terms of b_t and b_t^\dagger as

$$x_t = \sqrt{\frac{\hbar}{2m\omega}} (b_t + b_t^\dagger) \quad (39)$$

and has the correlation function

$$\langle 0|x_t x_s|0\rangle = \frac{\hbar}{2m\omega}\delta(t-s). \quad (40)$$

Thus the carrier space of our history theory is Gaussian white noise.

Finally, we note that the formalism discussed above can be extended to a wide class of Hamiltonian operators by employing the types of argument used by axiomatic field theorists to construct euclidean quantum field theories in one space plus one time dimension [9]. In particular, the underlying Gaussian stochastic process of our history theory of the simple harmonic oscillator can be successfully perturbed by the addition to H_t of a wide class of polynomial functions of the configuration variable x_t .

D. The ‘ n -particle’ History Propositions

The Fock-space construction produces a natural collection of history propositions: namely, those represented by the projection operators onto what, in a normal quantum field theory, would be called the ‘ n -particle states’. To see what these correspond to physically in our case we note first that a δ -function normalised basis for $\mathcal{F}[L^2(\mathbb{R}, dt)]$ is given by the vectors $|0\rangle, |t_1\rangle, |t_1, t_2\rangle, \dots$ where $|t_1\rangle := b_{t_1}^\dagger|0\rangle$, $|t_1, t_2\rangle := b_{t_1}^\dagger b_{t_2}^\dagger|0\rangle$, *etc* (of course, properly normalised vectors are of the form $|\phi\rangle := b_\phi^\dagger|0\rangle$ *etc* for suitable smearing function ϕ). The physical meaning of the projection operators of the form $|t\rangle\langle t|$ (or, more rigorously, $|\phi\rangle\langle\phi|$), $|t_1, t_2\rangle\langle t_1, t_2|$, *etc*, can be seen by studying the equations

$$H(\chi)|0\rangle = 0 \quad (41)$$

$$H(\chi)|t\rangle = \hbar\omega\chi(t)|t\rangle \quad (42)$$

$$H(\chi)|t_1, t_2\rangle = \hbar\omega[\chi(t_1) + \chi(t_2)]|t_1, t_2\rangle \quad (43)$$

or, in totally unsmeared form,

$$H_t|0\rangle = 0 \quad (44)$$

$$H_t|t_1\rangle = \hbar\omega\delta(t-t_1)|t_1\rangle \quad (45)$$

$$H_t|t_1, t_2\rangle = \hbar\omega[\delta(t-t_1) + \delta(t-t_2)]|t_1, t_2\rangle. \quad (46)$$

It is clear from the above that, for example, the projector $|t_1, t_2\rangle\langle t_1, t_2|$ represents the proposition that there is a unit of energy $\hbar\omega$ concentrated at the time point t_1 and another unit concentrated at the time point t_2 . Note that $H(\chi)|t, t\rangle = 2\hbar\omega\chi(t)|t, t\rangle$, and hence $|t, t\rangle\langle t, t|$ represents the proposition that there are two units of energy concentrated at the *single* time point t (thus exploiting the bose-structure of the canonical history algebra!). This interpretation of projectors like $|t_1, t_2\rangle\langle t_1, t_2|$ is substantiated by noting that the time-averaged energy obtained by choosing the averaging function χ to be 1 acts on these vectors as

$$\int_{-\infty}^{\infty} ds H_s|t\rangle = \hbar\omega|t\rangle \quad (47)$$

$$\int_{-\infty}^{\infty} ds H_s|t_1, t_2\rangle = 2\hbar\omega|t_1, t_2\rangle \quad (48)$$

and so on. This is the way in which the HPO account of the simple harmonic oscillator recovers the integer-spaced energy spectrum of standard quantum theory.

Finally, we note in passing that

$$\frac{1}{\hbar\omega} \int_{-\infty}^{\infty} ds s H_s |t_1, t_2, \dots, t_n\rangle = (t_1 + t_2 + \dots + t_n)|t_1, t_2, \dots, t_n\rangle \quad (49)$$

so that $\frac{1}{\hbar\omega} \int_{-\infty}^{\infty} ds s H_s$ acts as a ‘total time’ or ‘center-of-time’ operator.

E. The Heisenberg Picture

It is interesting to investigate the analogue of the Heisenberg picture in our continuous-time HPO theory. In standard quantum theory, the Heisenberg-picture version of an operator A is defined with respect to a time origin $t = 0$ as

$$A_H(s) := e^{isH/\hbar} A e^{-isH/\hbar}. \quad (50)$$

In particular, for the simple harmonic oscillator we have

$$x(s) = \cos[\omega s]x + \frac{1}{m\omega} \sin[\omega s]p \quad (51)$$

$$p(s) = -m\omega \sin[\omega s]x + \cos[\omega s]p. \quad (52)$$

The Heisenberg-picture operator $x(s)$ satisfies the classical equation of motion

$$\frac{d^2 x(s)}{ds^2} + \omega^2 x(s) = 0, \quad (53)$$

and the commutator of these operators is

$$[x(s_1), x(s_2)] = \frac{i\hbar}{m\omega} \sin[\omega(s_1 - s_2)] \quad (54)$$

which, on using the equation of motion

$$p := m \left. \frac{dx(s)}{ds} \right|_{s=0}, \quad (55)$$

reproduces the familiar canonical commutation relation Eq. (4).

In trying to repeat this construction for the history theory we might be tempted to define the Heisenberg-picture analogue of, say, x_t as

$$x_{H,t}(s) := e^{isH_t/\hbar} x_t e^{-isH_t/\hbar}. \quad (56)$$

However, this expression is not well-defined since it corresponds to choosing the test-function in Eq. (27) as $\chi(t') := s\delta(t - t')$, which leads to ill-defined products of $\delta(t - t')$.

What is naturally suggested instead is to define ‘time-averaged’ Heisenberg quantities

$$x_{\kappa,t} := e^{iH(\kappa)/\hbar} x_t e^{-iH(\kappa)/\hbar} = \cos[\omega\kappa(t)]x_t + \frac{1}{m\omega} \sin[\omega\kappa(t)]p_t \quad (57)$$

for suitable test functions κ . The analogue of the equation of motion Eq. (53) is the functional differential equation

$$\frac{\delta^2 x_{\kappa,t}}{\delta\kappa(s_1)\delta\kappa(s_2)} + \delta(t - s_1)\delta(t - s_2)\omega^2 x_{\kappa,t} = 0, \quad (58)$$

while the history analogue of Eq. (55) is

$$\delta(t - s)p_t = m \left. \frac{\delta x_{\kappa,t}}{\delta\kappa(s)} \right|_{\kappa=0}, \quad (59)$$

and the analogue of the ‘covariant commutator’ Eq. (54) is

$$[x_{\kappa_1,t_1}, x_{\kappa_2,t_2}] = \frac{i\hbar}{m\omega} \delta(t_1 - t_2) \sin[\omega(\kappa_1(t_1) - \kappa_2(t_2))] \quad (60)$$

which correctly reproduces the canonical history algebra.

It is worth remarking that we could have proceeded in a slightly different way by starting with a set of operators $x_t(s)$ that satisfy a postulated history version of the covariant commutator Eq. (54)

$$[x_{t_1}(s_1), x_{t_2}(s_2)] = \frac{i\hbar}{m\omega} \delta(t_1 - t_2) \sin[\omega(s_1 - s_2)] \quad (61)$$

and with the standard Schrödinger canonical history operators then being *defined* as $x_t := x_t(0)$ and $p_t := m \left. \frac{dx_t(s)}{ds} \right|_{s=0}$. However, in fact, this is just a special case of the first scheme with the test function κ chosen to be $\kappa(t) := s$ for all t ; *i.e.*, the ‘Heisenberg picture’ operators $x_t(s)$ are generated by the time-averaged energy in the form

$$x_t(s) := e^{is \int dr H_r/\hbar} x_t e^{-is \int dr H_r/\hbar}. \quad (62)$$

In our HPO formalism, the Heisenberg-picture operators—unlike those in the Schrödinger picture—have no obvious direct physical interpretation and their main use is likely to be mathematical. Therefore, there is no *a priori* reason for rejecting the simple time-averaged quantity in Eq. (62). What *is* clear however is that—whichever version is used—*two* different time labels appear: the ‘external’ label t that specifies the time at which a proposition is asserted, and the ‘internal’ label s that specifies the time parameter in the Heisenberg picture associated with the copy of standard quantum theory on the Hilbert space \mathcal{H}_t .

F. The Relation With Continuous Temporal Logic

Relating the construction above to the idea of ‘continuous temporal logic’ involves showing that

$$\otimes_{t \in \mathbb{R}} L_t^2(\mathbb{R}, dx) \simeq \mathcal{F}[L^2(\mathbb{R}, dt)] \quad (63)$$

where $L^2(\mathbb{R}, dx)$ is the Hilbert space of the standard quantum theory of a particle moving in one dimension. For completeness we shall summarise here the discussion of this relation given in our earlier paper [1].

At a heuristic level, the inner product between two continuous tensor product vectors $|\otimes_{t \in \mathbb{R}} u_t\rangle$ and $|\otimes_{t \in \mathbb{R}} v_t\rangle$ is required to be

$$\langle \otimes_{t \in \mathbb{R}} u_t | \otimes_{t \in \mathbb{R}} v_t \rangle_{\otimes_t \mathcal{H}_t} = \prod_{t \in \mathbb{R}} \langle u_t | v_t \rangle_{\mathcal{H}_t} := \exp \int_{-\infty}^{\infty} dt \log \langle u_t | v_t \rangle_{\mathcal{H}_t}. \quad (64)$$

Since $\langle u_t | v_t \rangle$ is a complex number the logarithm on the right hand side of Eq. (64) is generally not well-defined. However, consider the special case of trying to construct the continuous tensor product of copies of a *Fock* space $\mathcal{F}[K]$ for some Hilbert space K . In particular, consider the *coherent states* $|\exp \phi\rangle, |\phi\rangle \in K$, defined as

$$|\exp \phi\rangle := \oplus_{n=0}^{\infty} \frac{1}{n!} (\otimes |\phi\rangle)^n \quad (65)$$

where $(\otimes |\phi\rangle)^n$ denotes the tensor product of $|\phi\rangle$ with itself n times (and with the convention that $(|\phi\rangle)^0 := 1$). Then

$$\langle \exp \phi | \exp \psi \rangle_{\mathcal{F}[K]} = \exp \langle \phi | \psi \rangle_K \quad (66)$$

and hence, using Eq. (64),

$$\langle \otimes_t \exp \phi_t | \otimes_t \exp \psi_t \rangle_{\otimes_t \mathcal{F}[K]_t} = \exp \int_{-\infty}^{\infty} dt \langle \phi_t | \psi_t \rangle_{K_t} \quad (67)$$

which *is* well-defined. But the exponent in the right hand side of Eq. (67) is just the inner product in the direct integral $\int^{\oplus} K_t dt$ of the Hilbert spaces K_t , and hence we arrive at the basic isomorphism

$$\begin{aligned} \otimes_{t \in \mathbb{R}} \mathcal{F}[K]_t &\simeq \mathcal{F}[\int^{\oplus} K_t dt] \\ \otimes_t |\exp \phi_t\rangle &\mapsto |\exp \phi(\cdot)\rangle. \end{aligned} \quad (68)$$

However, the single-time Hilbert space of our theory— $L^2(\mathbb{R}, dx)$ —can be written as the Fock space for the one-dimensional Hilbert space \mathbb{C} via the isomorphism

$$\mathcal{F}[\mathbb{C}] \simeq L^2(\mathbb{R}, dx) \quad (69)$$

$$|\exp z\rangle \mapsto \langle x | \exp z \rangle := (2\pi)^{-1/4} e^{zx - (1/2)z^2 - (1/4)x^2} \quad (70)$$

where the right hand side involves the familiar coherent states in $L^2(\mathbb{R}, dx)$. Thus the isomorphism Eq. (68) becomes

$$\otimes_{t \in \mathbb{R}} L_t^2(\mathbb{R}, dx) \simeq \mathcal{F}[\int^{\oplus} \mathbb{C}_t dt]. \quad (71)$$

But the direct integral $\int^{\oplus} \mathbb{C}_t dt$ is isomorphic to $L^2(\mathbb{R}, dt)$ via the map that takes the parametrised family of complex numbers λ_t to the *function* $\lambda(\cdot)$ in $L^2(\mathbb{R}, dt)$ [i.e., $\lambda(t) := \lambda_t$]. Hence we arrive at the desired isomorphism in Eq. (63).

G. The Extension to Three Dimensions

The extension of the formalism above to a particle moving in three spatial dimensions appears at first sight to be unproblematic. The analogue of the history algebra Eqs. (8)–(10) is

$$[x_{t_1}^i, x_{t_2}^j] = 0 \quad (72)$$

$$[p_{t_1}^i, p_{t_2}^j] = 0 \quad (73)$$

$$[x_{t_1}^i, p_{t_2}^j] = i\hbar\delta^{ij}\delta(t_1 - t_2) \quad (74)$$

$i, j = 1, 2, 3$; while the formal expression Eq. (18) for the energy at time t becomes

$$H_t := \frac{\underline{p}_t \cdot \underline{p}_t}{2m} + \frac{m\omega^2}{2} \underline{x}_t \cdot \underline{x}_t. \quad (75)$$

It is straightforward to generalise the discussion above to this situation and, in particular, to find a Fock representation of Eqs. (72)–(74) in which the rigorous analogues of Eq. (75) exist as *bona fide* self-adjoint operators. However, an interesting issue then arises that has no analogue in one-dimensional quantum theory. Namely, we expect to have angular-momentum operators whose formal expression is

$$L_t^i := \epsilon^i_{jk} x_t^j p_t^k \quad (76)$$

and whose commutators can be computed heuristically as

$$[L_t^i, L_s^j] = i\hbar\epsilon^{ij}_k \delta(t - s) L_t^k. \quad (77)$$

Such operators L_t^i can be constructed rigorously using, for example, the method employed for the energy operators H_t : viz., compute the automorphisms of the canonical history algebra that are formally induced by the angular-momentum operators and then see if these automorphism can be unitarily implemented in the given Fock representation. However, the interesting observation is that, even if this can be done (which is the case, see below), this does not guarantee *a priori* that the commutators in Eq. (77) will be reproduced: in particular, it is necessary to check directly if a *c-number central extension* is present since we know from other branches of theoretical physics that algebras of the type in Eq. (77) are prone to such anomalies.

An obvious technique for evaluating such a commutator would be to define the angular momentum operators by point-splitting in the form

$$L_{t,\epsilon}^i := i\hbar\epsilon^i_{jk} (b_t^j)^\dagger b_{t+\epsilon}^k \quad (78)$$

so that the commutator in Eq. (77) is the analogue of an equal-time commutator in standard quantum field theory, and the point-splitting is the analogue of spatial point splitting. It is then straightforward to compute the commutators of these point-split operators and take the limit $\epsilon \rightarrow 0$. The result is the anticipated algebra Eq. (77).

However, in standard quantum field theory it is known that the limit of the commutator has to be considered at unequal times (*i.e.*, using Heisenberg-picture operators), and that there is a subtle relation between the two limits of the times becoming equal and the spatial point splitting tending to zero [10]. Therefore, in order to calculate correctly the commutator in our case it seems appropriate to consider the analogue of an unequal time commutator, namely

$$[L_{\chi,t,\epsilon}^i, L_{0,s,\epsilon}^j] \quad (79)$$

where

$$L_{\chi,t,\epsilon}^i := i\hbar\epsilon^i_{jk} (b_{\chi,t}^j)^\dagger b_{\chi,t+\epsilon}^k, \quad (80)$$

and where

$$b_{\chi,t}^k := e^{iH(\chi)} b_t^k e^{-iH(\chi)} = e^{-i\omega\chi(t)} b_t^k \quad (81)$$

is a time-averaged Heisenberg picture operator of the type defined earlier.

It is not difficult to show that

$$\begin{aligned} [L_{\chi,t,\epsilon}^i, L_{0,s,\epsilon}^j] &= -\hbar^2 e^{i\omega(\chi(t) - \chi(t+\epsilon))} \\ &\times [\delta(t - s + \epsilon) \left((b_t^j)^\dagger b_{t+2\epsilon}^i - \delta^{ij} (b_t^m)^\dagger b_{t+2\epsilon}^m \right) \\ &\quad - \delta(t - s - \epsilon) \left((b_{t-\epsilon}^i)^\dagger b_{t+\epsilon}^j - \delta^{ij} (b_{t-\epsilon}^m)^\dagger b_{t+\epsilon}^m \right)] \end{aligned} \quad (82)$$

and then, by evaluating the matrix element of the commutator in the vacuum state, one sees that there is no central extension in this case. Furthermore, by considering the matrix element of the commutator in general coherent states, one can check that the limits of $\epsilon \rightarrow 0$ and $\chi \rightarrow 0$ are straightforward, and that as long as the test functions are smooth, the angular momentum generators do indeed satisfy the heuristic commutator Eq. (77) in the limit.

H. The Role of the Velocity Operator

The HPO approach to the consistent-histories theory has the striking feature that, formally, there exists an operator that corresponds to propositions about the *velocity* of the system: namely, $\dot{x}_t := \frac{d}{dt}x_t$. More rigorously, we can adopt the procedure familiar from standard quantum field theory and define

$$\dot{x}_f := -x_{\dot{f}} \quad (83)$$

which is meaningful provided that (i) the test-function f is differentiable; and (ii) f ‘vanishes at infinity’ so that the implicit integration by parts used in Eq. (83) is allowed; *i.e.*, heuristically, $x_f = \int_{-\infty}^{\infty} dt x_t f_t$.

The rigorous existence of \dot{x}_t depends on the precise choice of test-function space used in the smeared form of the CHA in Eqs. (14)–(16). In the analogous situation in normal quantum field theory, the test-functions are chosen so that the spatial derivatives of the quantum field exist, this being necessary to define the Hamiltonian operator. In our case, the situation is somewhat different since the energy operator H_t [see Eq. (18)] does not depend on \dot{x}_t and hence there is no *a priori* requirement for \dot{x}_t to exist. However, what *is* clear from Eq. (8) is that *if* \dot{x}_t exists then

$$[x_t, \dot{x}_s] = 0 \quad (84)$$

and hence our theory allows for history propositions that include assertions about the position of the particle and its velocity at the same time; in particular, the velocity \dot{x}_t and momentum p_t are not related. In this context it should be emphasised once more that x_t , $t \in \mathbb{R}$, is a one-parameter family of *Schrödinger*-picture operators—it is *not* a Heisenberg-picture operator, and the equations of motion do not enter at this level.

The existence of a velocity operator that commutes with position is a striking property of the HPO approach to consistent histories and raises some intriguing questions. For example, a classic paper by Park and Margenau [11] contains an interesting discussion of the uncertainty relations, including a claim that it *is* possible to measure position and momentum simultaneously provided the latter is defined using time-of-flight measurements. The existence in our formalism of the vanishing commutator Eq. (84) throws some new light on this old discussion. Also relevant in this respect is Hartle’s discussion of the operational meaning of momentum in a history theory [12]. In particular, he emphasises that an accurate measurement of momentum requires a long time-of-flight, whereas—on the other hand—our definition of velocity as the time-derivative of the history variable x_t clearly involves a vanishingly small time interval. Presumably this is the operational difference between momentum and velocity in the HPO approach to consistent histories.

The potential existence of \dot{x}_t also raises the interesting possibility of defining a ‘velocity-extended’ version of the energy H_t as

$$H_t := \frac{p_t^2}{2m} + \frac{m}{2}(\omega^2 x_t^2 + \lambda \dot{x}_t^2) \quad (85)$$

for some real parameter $\lambda \geq 0$. Note that in the one-dimensional quantum field theory analogue in which x_t and p_t are replaced by $\phi(x)$ and $\pi(x)$ respectively, for an appropriate choice of λ the expression in Eq. (85) becomes the usual Hamiltonian density $H(x)$ for a massive scalar field (the $\lambda = 0$ case of Eq. (18) then corresponds to an *ultralocal* quantum field). Guided by this observation, we can try to repeat our earlier analysis and look for a representation of the canonical history algebra in which a suitably smeared version of this new energy exists as a proper operator. The quantum field theory analogue suggests that the appropriate replacement for Eq. (29) is

$$c_t := \left(\frac{m\sqrt{\omega^2 - \lambda D^2}}{2\hbar} \right)^{1/2} x_t + i \left(\frac{1}{2m\hbar\sqrt{\omega^2 - \lambda D^2}} \right)^{1/2} p_t \quad (86)$$

where D denotes the differential operator $\frac{d}{dt}$. Note that $-D^2$ is a positive semi-definite operator on $L^2(\mathbb{R}, dt)$, and hence the square-root of $\omega^2 - \lambda D^2$ is well-defined; of course, to make all this rigorous a suitably-smeared form of Eq. (86) should be used.

We note that

$$[c_t, c_s^\dagger] = \delta(t - s) \quad (87)$$

and that, *if* it existed, the smeared form $H(\chi)$ of the velocity-extended Hamiltonian would generate the CHA automorphism

$$e^{iH(\chi)/\hbar} c_t e^{-iH(\chi)/\hbar} = \exp[-i\chi(t)\sqrt{\omega^2 - \lambda D^2}] c_t. \quad (88)$$

For the special case in which χ is a *constant*, the right hand side corresponds to a unitary transformation on the ‘one-particle’ space $L^2(\mathbb{R}, dt)$, and hence the time-averaged energy $\int_{-\infty}^{\infty} dt H_t$ exists as a genuine operator on the associated Fock space. Of course, the eigenvalues and eigenvectors of this operator are well-known for the quantum field theory analogue of the total Hamiltonian operator $\int dx H(x)$; in the history theory, the associated projection operators correspond to appropriate propositions about the value of the time average of the energy.

Note that the relation between x_t and the new annihilation and creation operators is non-local in time:

$$x_t = \left(\frac{\hbar}{2m\sqrt{\omega^2 - \lambda D^2}} \right)^{1/2} (c_t + c_t^\dagger). \quad (89)$$

In particular, the correlation function is

$$\langle 0 | x_t x_s | 0 \rangle = \left(\frac{\hbar}{2m\sqrt{\omega^2 - \lambda D^2}} \right) \delta(t - s) \quad (90)$$

where the non-local quantity on the right hand side is the Green’s function of the elliptic, partial differential operator $\sqrt{\omega^2 - \lambda D^2}$. Thus we still have a Gaussian stochastic process but it is ‘softer’ than the one constructed earlier whose correlation function was Eq. (40). Of course, existence of non-local terms is a common occurrence in normal relativistic quantum field theory, but there the non-locality is in space, not time. It remains to be seen whether the ‘velocity-extended’ Hamiltonian in Eq. (85) has any real physical application in the consistent histories theory.

III. QUANTUM FIELD THEORY

A. The canonical history algebra

We wish now to extend the discussion to the HPO theory of a free scalar field. Hartle [13] proposed a consistent histories approach to quantum field theory based on path integrals, and Blencowe [14] gave a careful analysis of the use of class operators. However, almost nothing has been said about the HPO scheme in this context, and we shall now briefly present the necessary developments. The resemblance of the history version of quantum mechanics (‘field theory in zero spatial dimensions’) to a canonical field theory in one spatial dimension suggests that the history version of quantum field theory in three spatial dimensions should resemble canonical quantum field theory in *four* spatial dimensions. We shall see that this expectation is fully justified.

The first step in constructing an HPO version of quantum field theory is to foliate four-dimensional Minkowski space-time with the aid of a time-like vector n^μ that is normalised by $\eta_{\mu\nu} n^\mu n^\nu = 1$, where the signature of the Minkowski metric $\eta_{\mu\nu}$ has been chosen as $(+, -, -, -)$. The canonical commutation relations for a standard bosonic quantum field theory (the analogue of Eq. (4)) in three spatial dimensions are

$$[\phi(\underline{x}_1), \phi(\underline{x}_2)] = 0 \quad (91)$$

$$[\pi(\underline{x}_1), \pi(\underline{x}_2)] = 0 \quad (92)$$

$$[\phi(\underline{x}_1), \pi(\underline{x}_2)] = i\hbar\delta^3(\underline{x}_1 - \underline{x}_2) \quad (93)$$

where \underline{x}_1 and \underline{x}_2 are three-vectors that are spatial with respect to the foliation vector n . In constructing the associated HPO theory we shall assume that the passage from the canonical algebra Eq. (4) to the history algebra Eqs. (8)–(10) is reflected in the field theory case by passing from Eqs. (91)–(93) to

$$[\phi_{t_1}(\underline{x}_1), \phi_{t_2}(\underline{x}_2)] = 0 \quad (94)$$

$$[\pi_{t_1}(\underline{x}_1), \pi_{t_2}(\underline{x}_2)] = 0 \quad (95)$$

$$[\phi_{t_1}(\underline{x}_1), \pi_{t_2}(\underline{x}_2)] = i\hbar\delta(t_1 - t_2)\delta^3(\underline{x}_1 - \underline{x}_2) \quad (96)$$

where, for each $t \in \mathbb{R}$, the fields $\phi_t(\underline{x})$ and $\pi_t(\underline{x})$ are associated with the spacelike hypersurface (n, t) whose normal vector is n and whose foliation parameter is t ; in particular, the three-vector \underline{x} in $\phi_t(\underline{x})$ or $\pi_t(\underline{x})$ denotes a vector in this space.

In using this algebra, we have in mind a representation that is some type of continuous tensor product $\otimes_{t \in \mathbb{R}} \mathcal{H}_t$ where each \mathcal{H}_t carries a representation of the standard canonical commutation relations Eqs. (91)–(93) for a scalar field theory associated with the given spacetime foliation. However, to emphasise the underlying spacetime picture it is convenient to rewrite Eqs. (94)–(96) in terms of four-vectors X and Y as

$$[\phi(X), \phi(Y)] = 0 \quad (97)$$

$$[\pi(X), \pi(Y)] = 0 \quad (98)$$

$$[\phi(X), \pi(Y)] = i\hbar\delta^4(X - Y). \quad (99)$$

In relating these expressions to those in Eqs. (94)–(96) the three-vector \underline{x} may be equated with a four-vector x_n that satisfies $n \cdot x_n = 0$ (the dot product is taken with respect to the Minkowski metric $\eta_{\mu\nu}$) so that the pair $(t, \underline{x}) \in \mathbb{R} \times \mathbb{R}^3$ is associated with the spacetime point $X = tn + x_n$ (in particular, $t = n \cdot X$). Note, however, that the covariant-looking nature of these expressions is deceptive and it is not correct to assume *a priori* that the fields $\phi(X)$ and $\pi(Y)$ transform as spacetime scalars under the action of some ‘external’ spacetime Poincaré group that acts on the X and Y labels—as things stand there is an implicit n label on both ϕ and π . We shall return to this question later.

B. The Hamiltonian Algebra

The key idea of our HPO approach to quantum field theory is that the physically-relevant representation of the canonical history algebra Eqs. (94)–(96) [or, equivalently, Eqs. (97)–(99)] is to be selected by requiring the existence of operators that represent history propositions about temporal averages of the energy defined with respect to the chosen spacetime foliation. Thus, for a fixed foliation vector n , we seek a family of ‘internal’ Hamiltonians $H_{n,t}$, $t \in \mathbb{R}$, whose explicit formal form (*i.e.*, the analogue of Eq. (18)) can be deduced from the standard quantum field theory expression to be

$$H_{n,t} := \frac{1}{2} \int d^4X \{ \pi(X)^2 + (n^\mu n^\nu - \eta^{\mu\nu}) \partial_\mu \phi(X) \partial_\nu \phi(X) + m^2 \phi(X)^2 \} \delta(t - n \cdot X). \quad (100)$$

The analogous, temporally-averaged object is

$$\begin{aligned} H_n(\chi) &:= \int_{-\infty}^{\infty} dt \chi(t) H_{n,t} \\ &= \frac{1}{2} \int d^4X \{ \pi(X)^2 + (n^\mu n^\nu - \eta^{\mu\nu}) \partial_\mu \phi(X) \partial_\nu \phi(X) + m^2 \phi(X)^2 \} \chi(n \cdot X) \end{aligned} \quad (101)$$

where χ is a real-valued test function.

As in the discussion above of the simple harmonic oscillator, the next step is to consider the commutator algebra that would be satisfied by the operators $H_n(\chi)$ if they existed. These field-theoretic analogues of Eqs. (24)–(26) are readily computed as

$$[H_n(\chi), \phi(X)] = -i\hbar\chi(n \cdot X)\pi(X) \quad (102)$$

$$[H_n(\chi), \pi(X)] = i\hbar\chi(n \cdot X)K_n\phi(X) \quad (103)$$

$$[H_n(\chi_1), H_n(\chi_2)] = 0 \quad (104)$$

where K_n denotes the partial differential operator

$$(K_n f)(X) := [(\eta^{\mu\nu} - n^\mu n^\nu) \partial_\mu \partial_\nu + m^2] f(X). \quad (105)$$

The exponentiated form of Eqs. (102)–(103) is

$$e^{iH_n(\chi)/\hbar} \phi(X) e^{-iH_n(\chi)/\hbar} = \cos \left[\chi(n \cdot X) \sqrt{K_n} \right] \phi(X) + \frac{1}{\sqrt{K_n}} \sin \left[\chi(n \cdot X) \sqrt{K_n} \right] \pi(X) \quad (106)$$

$$e^{iH_n(\chi)/\hbar} \pi(X) e^{-iH_n(\chi)/\hbar} = -\sqrt{K_n} \sin \left[\chi(n \cdot X) \sqrt{K_n} \right] \phi(X) + \cos \left[\chi(n \cdot X) \sqrt{K_n} \right] \pi(X) \quad (107)$$

where the square-root operator $\sqrt{K_n}$, and functions thereof, can be defined rigorously using the spectral theory of the self-adjoint, partial differential operator K_n on the Hilbert space $L^2(\mathbb{R}^4, d^4X)$. Note that the expression $\chi(n \cdot X) \sqrt{K_n}$ is unambiguous since, viewed as an operator on $L^2(\mathbb{R}^4, d^4X)$, multiplication by $\chi(n \cdot X)$ commutes with K_n .

C. The Fock Space Representation

The right hand side of Eqs. (106)–(107) defines an automorphism of the CHA Eqs. (97)–(99) and the task is to find a representation of the latter in which these automorphisms are unitarily implemented. To this end, define new operators

$$q(X) := K_n^{1/4} \phi(X) \quad (108)$$

$$p(X) := K_n^{-1/4} \pi(X) \quad (109)$$

and

$$b(X) := \frac{1}{\sqrt{2}} \left(q(X) + ip(X) \right) = \frac{1}{\sqrt{2}} \left(K_n^{1/4} \phi(X) + i K_n^{-1/4} \pi(X) \right) \quad (110)$$

which satisfy

$$[b(X), b(Y)] = 0 \quad (111)$$

$$[b^\dagger(X), b^\dagger(Y)] = 0 \quad (112)$$

$$[b(X), b^\dagger(Y)] = \hbar \delta^4(X - Y). \quad (113)$$

Then

$$e^{iH_n(\chi)/\hbar} q(X) e^{-iH_n(\chi)/\hbar} = \cos \left[\chi(n \cdot X) \sqrt{K_n} \right] q(X) + \sin \left[\chi(n \cdot X) \sqrt{K_n} \right] p(X) \quad (114)$$

$$e^{iH_n(\chi)/\hbar} p(X) e^{-iH_n(\chi)/\hbar} = -\sin \left[\chi(n \cdot X) \sqrt{K_n} \right] q(X) + \cos \left[\chi(n \cdot X) \sqrt{K_n} \right] p(X) \quad (115)$$

and so

$$e^{iH_n(\chi)/\hbar} b(X) e^{-iH_n(\chi)/\hbar} = e^{-i\chi(n \cdot X) \sqrt{K_n}} b(X). \quad (116)$$

However, the operator defined on $L^2(\mathbb{R}^4)$ by

$$(U(\chi)\psi)(X) := e^{-i\chi(n \cdot X) \sqrt{K_n}} \psi(X) \quad (117)$$

is unitary, and hence—using the same type of argument invoked earlier for the simple harmonic oscillator—we conclude that the desired quantities $H_n(\chi)$ exist as self-adjoint operators on the Fock space $\mathcal{F}[L^2(\mathbb{R}^4, d^4X)]$ associated with the creation and annihilation operators $b^\dagger(X)$ and $b(X)$. The spectral projectors of these operators then represent propositions about the time-averaged value of the energy in the spacetime foliation determined by n .

D. The Question of External Lorentz Invariance

An important part of standard quantum field theory is a proof of invariance under the Poincaré group—something that, in the canonical formalism, is not totally trivial since the Schrödinger-picture fields depend on the reference frame (*i.e.*, the spacetime foliation). The key ingredient is a construction of the generators of the Poincaré group as explicit functions of the canonical field variables; in practice, the first step is often to construct the Heisenberg-picture fields with the aid of the Hamiltonian, and then to demonstrate manifest Poincaré covariance within that framework. The canonical fields associated with any spacelike surface in a particular Lorentz frame can then be obtained by restricting the Heisenberg fields (and their normal derivatives) to the surface.

When considering the role of the Poincaré group in the HPO picture of consistent histories, the starting point is the observation that, heuristically speaking, for a given foliation vector n —and for each value of the associated time t —there will be a Hilbert space \mathcal{H}_t carrying an independent copy of the standard quantum field theory. In particular, therefore, for fixed n , there will be a representation of the Poincaré group associated with each spacelike slice (n, t) , $t \in \mathbb{R}$. Thus if A_a , $a = 1, 2, \dots, 10$ denote the generators of the Poincaré group, there should exist a family of operators A_t^a which, for each $t \in \mathbb{R}$, generate the ‘*internal*’ Poincaré group $\mathcal{P}_{n,t}$ associated with the slice (n, t) . These operators will satisfy a ‘temporally gauged’ version of the Poincaré algebra. More precisely, if C^{ab}_c are the structure constants of the Poincaré group, so that

$$[A^a, A^b] = i C^{ab}_c A^c, \quad (118)$$

then the algebra satisfied by the history theory operators A_t^a is

$$[A_t^a, A_s^b] = i\delta(t-s)C^{ab}{}_c A_t^c \quad (119)$$

which, of course, reflects the way in which the canonical commutation relations Eqs. (91–93) are replaced by Eqs. (94–96) in the history theory.

As always in quantum theory, the energy operator is of particular importance, and in the present case we have a family of Hamiltonian operators $H_{n,t}$, $t \in \mathbb{R}$, which are related to the generators $P_{n,t}^\mu$ of translations for the quantum field theory associated with the hypersurface (n, t) by

$$H_{n,t} = n_\mu P_{n,t}^\mu. \quad (120)$$

In fact, it is straightforward to show that

$$P_{n,t}^\mu = n^\mu H_{n,t} + \int d^4X \delta(t - n \cdot X)(n^\mu n \cdot \partial\phi - \partial^\mu\phi)\pi \quad (121)$$

which suggests that, as would be expected, the components of $P_{n,t}^\mu$ normal to n act as the generators of spatial translations in the hypersurface (n, t) . Indeed, Eq. (102) generalises to

$$[P_n^\mu(\chi), \phi(X)] = -i\hbar\chi(n \cdot X)\{n_\mu\pi(X) + (\partial_\mu\phi(X) - n_\mu n \cdot \partial\phi(X))\}. \quad (122)$$

Similarly, the ‘temporally gauged’ Lorentz generators satisfy

$$[J_{n,t}^{\mu\nu}, \phi(X)] = i\hbar\delta(t - n \cdot X)\{X^\mu(\partial^\nu\phi - n^\nu n \cdot \partial\phi) - X^\nu(\partial^\mu\phi - n^\mu n \cdot \partial\phi) - (X^\mu n^\nu - X^\nu n^\mu)\pi\}. \quad (123)$$

As emphasised above, each generator of the group $\mathcal{P}_{n,t}$ acts ‘internally’ in the Hilbert space \mathcal{H}_t ; in particular, this is true of the Hamiltonian, which (modulo the need to smear in t) generates translations along an ‘internal’ time label s that is to be associated with each leaf (n, t) of the foliation. It is important to note that $H_{n,t}$ does *not* generate translations along the ‘external’ time parameter t that appears in the CHA Eqs. (94–96) and which labels the spacelike surface (of course, there is an analogous statement for the Hamiltonians H_t in the HPO model of the simple harmonic oscillator considered earlier). The existence of these internal Poincaré groups is sufficient to guarantee covariance of physical quantities, such as transition amplitudes, that can be calculated in the class operator version of the theory.

However, the HPO formalism admits an additional type of Poincaré group—what we shall call the ‘*external*’ Poincaré group—which is defined to act on the pair of labels (\underline{x}, t) that appear in the CHA Eqs. (94–96). Thus these labels include the ‘external’ time parameter t that specifies the leaf (n, t) of the foliation associated with the timelike vector n . In the context of the covariant-looking version Eqs. (97–99) of the CHA, the main question is whether the fields $\phi(X)$ and $\pi(X)$ transform in a covariant way under this external group.

As far as the field $\phi(X)$ is concerned it seems reasonable to consider the possibility that this may be an external scalar in the sense that there exists a unitary representation $U(\Lambda)$ of the external Lorentz group $U(\Lambda)$ such that

$$U(\Lambda)\phi(X)U(\Lambda)^{-1} = \phi(\Lambda X). \quad (124)$$

The spectral projectors of the (suitably smeared) operators $\phi(X)$ then represent propositions about the values of the spacetime field in a covariant way.

However, the situation for the field momentum $\pi(X)$ is different since this is intrinsically associated with the timelike vector n . Indeed, the natural thing would be to require the existence of a *family* of operators $\pi_n(X)$ where n lies in the hyperboloid of all timelike (future-pointing) vectors, and such that

$$U(\Lambda)\pi_n(X)U(\Lambda)^{-1} = \pi_{\Lambda n}(\Lambda X). \quad (125)$$

The next step in demonstrating external Poincaré covariance would be to extend the algebra (97–99) to include the n parameter on the π field; in particular, one would need to specify the commutator $[\pi_n(X), \pi_m(Y)]$, but it is not obvious *a priori* what this should be.

Another possibility would be to try to combine the Heisenberg picture—and its associated ‘internal’ time s —with the external time parameter t of the spacetime foliation to give some scheme that was manifestly covariant in the context of a five-dimensional space with signature $(+, +, +, -)$ associated with the variables (\underline{x}, t, s) . However, we do not know if this is possible and the demonstration of external Poincaré covariance, if it exists, remains the subject for future research.

IV. CONCLUSIONS

We have discussed the introduction of continuous-time histories within the ‘HPO’ version of the consistent-histories formalism in which propositions about histories of the system are represented by projection operators on a ‘history’ Hilbert space. The history algebra (whose representations specify this space) for a particle moving in one dimension is isomorphic to the canonical commutation relations for a one-dimensional quantum field theory, thus allowing the history theory to be studied using techniques drawn from quantum field theory. In particular, we have shown how the problem of the existence of infinitely many inequivalent representations of the history algebra can be solved by requiring the existence of operators whose spectral projectors represent propositions about time-averages of the energy.

We have shown how the Heisenberg picture is changed in the HPO formalism in such a way that the familiar, partial-differential equations of motion are replaced by functional differential equations; these operators are used in the proof that the angular momentum operators of the three-dimensional theory are anomaly free. The question of potential anomalies in the history algebra is rather intriguing and it would be interesting to study a theory in which such things might be expected, such as a fermionic system.

A striking property of the HPO formalism is the potential existence of a velocity operator that commutes with the position operators, thus opening up a new perspective on the old debate about the operational meaning of the Heisenberg uncertainty relations between position and momentum. The introduction of the velocity operator suggests a number of topics for future work: for example, it would be most interesting to see if some thing like the action functional of the classical action has a natural role to play in the HPO theory.

Finally, we have shown how the HPO scheme can be extended to the history version of canonical quantum field theory. We discussed the difference between the ‘internal’ and ‘external’ Poincaré groups and indicated how the former are implemented in the formalism. A major challenge for future research is to construct an HPO quantum field theory which is manifestly covariant under this external symmetry group.

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